

Integration

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Recall that in the previous section, we develop a method to compute the area under a curve $y = f(x)$. In particular, we used $\lim_{n \rightarrow \infty} R_n$ and $\lim_{n \rightarrow \infty} L_n$.

This method can be generalized. First, after dividing $[a, b]$ into n subintervals of size $\frac{b-a}{n}$, instead of choosing the right or left endpoint of each subinterval, we can actually choose any point x_i^* in $[x_{i-1}, x_i]$, so

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$$

Now we are going to generalize it further: f is a function defined on $[a, b]$ (now we do not assume f being continuous or positive) We divide $[a, b]$ into n smaller subintervals $a = x_0 < x_1 < \dots < x_n = b$ (now we do not assume the subintervals are of equal length) We say $[x_0, x_1], [x_1, x_2], \dots, [x_{n-1}, x_n]$ form a partition of $[a, b]$. For each $[x_{i-1}, x_i]$, let $\Delta x_i = x_i - x_{i-1}$, which is the length of this interval. We choose sample points x_i^* from $[x_{i-1}, x_i]$, and define a **Riemann Sum** associated with f and a partition of $[a, b]$ to be

$$\sum_{i=1}^n f(x_i^*) \Delta x_i$$

Geometrically, it corresponds to figure

when $f(x_i^*) < 0$, we see $f(x_i^*)$ is the negative of the corresponding area of the rectangle.

Observe that although we didn't cut the intervals in an evenly manner, as long as all the subintervals are very small, the region covered by these rectangles is a good approximation of the region bounded by $y = f(x)$ on $[a, b]$. If we define the area above x -axis to be positive and below to be negative, we see $\sum_{i=1}^n f(x_i^*) \Delta x_i$ is a good approximation of the area of the region bounded by $y = f(x)$ and $[a, b]$. This motivates the following definition of definite integrals.

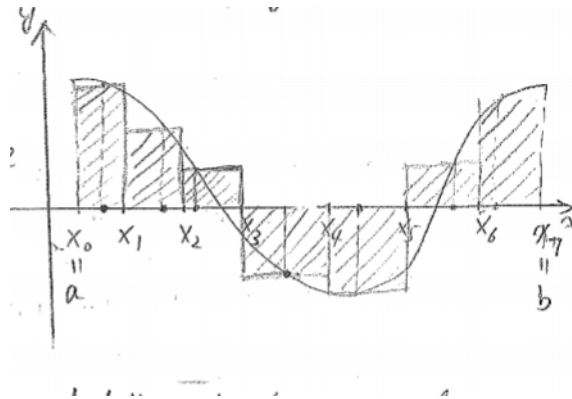


Figure 1: Riemann Sum

Definition 1. If f is a function defined on $[a, b]$, the **definite integral** of f from a to b is the number

$$\int_a^b f(x) dx = \lim_{\max \Delta x_i \rightarrow 0} \sum_{i=1}^n f(x_i^*) \Delta x_i$$

provided that this limit exists. If it does exist, we say f is integrable on $[a, b]$. f is called the integrand, and the process of calculating an integral is called integration.

The definite integral represents the area of the region between $y = f(x)$ and x -axis, with the convention that the area is positive above x -axis and negative below x -axis.

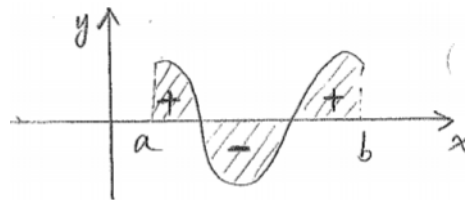


Figure 2: Definite Integral

Theorem 2. If f is continuous on $[a, b]$, or f only has finitely many jumping discontinuity points, then f is integrable on $[a, b]$.

Recall that in the definition of Riemann Sum, there is no restriction on the subdivision of $[a, b]$ and the choice of x_i^* , as long as when we take $\max \Delta x_i \rightarrow 0$, the Riemann Sum will be the definite integral $\int_a^b f(x) dx$. So we may take a good choice of Riemann Sum that is convenient for computation.

Example 3. Evaluate $\int_0^3 (x^3 - 6x) dx$. (We may need the identity $\sum_{k=1}^n k^3 = (\frac{n(n+1)}{2})^2$)

We may divide $[0, 3]$ into n subintervals of size $\frac{3}{n}$ with endpoints $x_0 = 0, x_1 = \frac{3}{n}, x_2 = \frac{6}{n}, \dots, x_i = \frac{3i}{n}, \dots, x_n = 3$. We see as $n \rightarrow \infty$, all $\Delta x_i \rightarrow 0$, $\max \Delta x_i \rightarrow 0$. We take $x_i^* = x_i = \frac{3i}{n}$, then

$$\begin{aligned} \int_0^3 f(x) dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x_i \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\left(\frac{3i}{n} \right)^3 - 6 \times \frac{3i}{n} \right] \frac{3}{n} \\ &= \lim_{n \rightarrow \infty} \frac{81}{n^4} \sum_{i=1}^n i^3 - \lim_{n \rightarrow \infty} \frac{54}{n^2} \sum_{i=1}^n i^2 \\ &= \lim_{n \rightarrow \infty} \frac{81}{n^4} \left(\frac{n(n+1)}{2} \right)^2 - \lim_{n \rightarrow \infty} \frac{54}{n^2} \frac{n(n+1)}{2} \\ &= \lim_{n \rightarrow \infty} \frac{81}{4} \left(\frac{n+1}{n} \right)^2 - \lim_{n \rightarrow \infty} 27 \times \frac{n+1}{n} \\ &= -\frac{27}{4} \end{aligned}$$

Proposition 4. (Midpoint Rule)

$$\int_a^b f(x) dx \approx \sum_{i=1}^n f(\bar{x}_i) \Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $\bar{x}_i = \frac{x_{i-1} + x_i}{2}$

Example 5. Use the midpoint rule with $n = 5$ to approximate $\int_1^2 \frac{1}{x} dx$

$\Delta x = \frac{2-1}{5} = 0.2$, the five subintervals are $[1, 1.2], [1.2, 1.4], [1.4, 1.6], [1.6, 1.8], [1.8, 2]$, and the midpoints are 1.1, 1.3, 1.5, 1.7, 1.9

$$\begin{aligned} \int_1^2 \frac{1}{x} dx &= (f(1.1) + f(1.3) + f(1.5) + f(1.7) + f(1.9)) \times 0.2 \\ &= \left(\frac{1}{1.1} + \frac{1}{1.1} + \frac{1}{1.1} + \frac{1}{1.1} + \frac{1}{1.1} \right) \times 0.2 \\ &\approx 0.691908 \end{aligned}$$

Definition 6. If $a > b$, we define $\int_a^b f(x) dx = -\int_b^a f(x) dx$. Also we define $\int_a^a f(x) dx = 0$

Proposition 7. Suppose the following integrals exist, then:

1. $\int_a^b C dx = C(b - a)$, where C is a constant.
2. $\int_a^b f(x) \pm g(x) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$
3. $\int_a^b C f(x) dx = C \int_a^b f(x) dx$, where C is a constant
4. If $f(x) \geq g(x)$ on $[a, b]$, then $\int_a^b f(x) dx \geq \int_a^b g(x) dx$
5. If $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x) dx \geq 0$
6. If $m \leq f(x) \leq M$ on $[a, b]$, then $m(b - a) \leq \int_a^b f(x) dx \leq M(b - a)$
7. $\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$
8. $\int_a^d f(x) dx - \int_b^c f(x) dx = \int_a^b f(x) dx + \int_c^d f(x) dx$

Example 8. Consider $\int_0^1 e^{-x^2} dx$. We know on $[0, 1]$, $e^{-1} \leq e^{-x^2} \leq e^0 = 1$, so

$$\frac{1}{e} = e^{-1}(1 - 0) \leq \int_0^1 e^{-x^2} dx \leq 1 \times (1 - 0) = 1$$